Patterns in Pascal’s Triangle

When numbers appear in arrays (a rectangular arrangement of objects in rows and columns), is it possible to identify a pattern that enables us to continue the array indefinitely? One of the most interesting number patterns is Pascal’s triangle, named for Blaise Pascal, a famous French mathematician and philosopher. In the following activities, we will explore how Pascal’s triangle is formed and discover many of its patterns.

Creating Pascal’s Triangle

1. Obtain a blank copy of the Pascal’s triangle activity sheet.

Fill in the cells of the first five rows of Pascal’s triangle as follows:

Row 0: 1
Row 1: 1, 1
Row 2: 1, 2, 1
Row 3: 1, 3, 3, 1
Row 4: 1, 4, 6, 4, 1

a. Study the array of numbers. What patterns do you see in the arrangement of numbers?

Discuss your findings with others in your group or class.

b. Predict the entries for the cells in the next row of numbers. How did you determine the entries?

c. Fill in the cells for the remaining rows of Pascal’s triangle.

Now that we know how to construct Pascal’s triangle, let’s look a little deeper at its rows and diagonals to find some of its hidden patterns.

The Sums of the Rows

One of the patterns of Pascal’s triangle is displayed when you find the sums of the elements in the rows. Any row starting with the number 1 can be added horizontally (to find the sum).

3. Find the sum of each row for rows 0–4 of Pascal’s triangle. Remember, at the tip of the triangle is the number 1, which is row 0.

Table 1

<table>
<thead>
<tr>
<th>Row</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

4. What patterns do you notice in the horizontal sums?
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5. Check your conjecture by finding the sum of the numbers in the following rows:

Row 5:

Row 6:

Row 7:

Does the pattern you predicted continue? Discuss your findings with others in your group or class.

6. How would you find the sum of the numbers in the tenth row of Pascal’s triangle?

7. How would you find the sum of the numbers in the $n$th row of Pascal’s triangle?

Summing Up Pascal (Sum More)

8. Using table 2 below, find the total sum of all the numbers in Pascal’s triangle up through row 8. Begin by adding the number 1 in the first row of Pascal’s triangle to the sum of the numbers in the second row. Then add that sum to the sum of the numbers in the third row. Record your results in table 2.

9. Use your pattern to describe how to find the sum of all the numbers in a Pascal’s triangle with fifty rows, then with $n$ rows.

Pascal’s Petals

Choose three colored pencils to work with.

10. Select any cell in Pascal’s triangle that is surrounded by six other cells. Shade this cell using your first color. We will call this cell the center of the design.

Example:

Starting with the petal above and to the left of the center, use your second color to fill in the three alternating petals that surround the center cell.

Example:

Table 2

<table>
<thead>
<tr>
<th>Row Number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>50</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of All Numbers in the Triangle</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What pattern do you notice? Record your observations below, then discuss them with others in your group or class.
Use the third color to fill in the three remaining petals around the chosen center.

Example:

11. What is the product of the numbers in the first set of (the yellow) petals?

12. What is the product of the numbers in the second set of (the orange) petals?

13. What do you notice about the products?

Discuss your observations with others in your group or class. Can you come up with an explanation for why this works?

14. Write the prime factorization of each of the numbers in the yellow petals. Then write the product of the three prime factorizations:

15. Write the prime factorization of each of the numbers in the orange petals. Then write the product of the three prime factorizations:


17. Do you think you would get the same result if you picked a different starting cell? Why, or why not?

Choose a different cell as the center of your petal. Repeat the shading described in question 10.

18. Write the prime factorization for the yellow petals and for the orange petals. Then write the product of the prime factorizations.

19. Compare the products of the two prime factorizations. What do you notice?
20. What can you conclude about the petals surrounding a cell in Pascal’s triangle?

21. When you divide any number by two, what possible remainders can you get?

The remainders you get when dividing by any number are called modular numbers, or mod for short. When we want to consider the remainder when dividing a number \( n \) by two, we write, “\( n \mod 2 \).”

22. Determine 14 mod 2 _____
    81 mod 2 _____
    1001 mod 2 _____
    2456 mod 2 _____

What conclusions can you make about the numbers that have zero as a remainder and those that have one as a remainder when dividing by two?

23. Using a completed Pascal’s triangle that goes to row 15, shade all the odd numbers. Use modular number notation to write an equation showing which numbers were shaded.

24. Record any patterns that you notice.


Enter 50 for the number of rows and 3 for the mod number, but do not click the Enter key yet.

The Web site reads, “In Z mod \( n \) you will have \( n \) colors/group elements.” What do you think this means?
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a. Describe what each color in the triangle represents. Then write a modular number equation for each.

b. Now let’s revisit the statement, “In Z mod n you will have n colors/group elements.” What do you think this means?

c. Clear the triangle by pressing Back. Choose another value for the mod, and display the triangle. How does this pattern compare with the previous pattern?

Binomial Coefficients, Part 1

Let’s do a quick algebra review. A binomial is a polynomial expression with two terms, such as $a + b$, $a^2 + 1$ or $a^4 - 3b$. 

**Binomial expansion** refers to a formula by which one can expand, or spread out, expressions like $(a + b)^n$, where the entire binomial is raised to a power. To show how to raise binomials to positive integer powers without doing the actual multiplication, consider the following binomial expansions:

- $(a + b)^0 = 1$  
  1 term
- $(a + b)^1 = (a + b)$  
  2 terms
- $(a + b)^2 = a^2 + 2ab + b^2$  
  3 terms
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$  
  4 terms
- $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$  
  5 terms

26. State two patterns that appear in these expansions.

27. Let’s examine the connections between Pascal’s triangle and the coefficients of the terms in the binomial expansion.

a. Begin by comparing the binomial expansion for an exponent of 0 to row 0 of Pascal’s triangle.

b. Compare the exponents of the terms for a binomial expansion with power of 1 to row 1 of Pascal’s triangle.

c. Compare the exponents of the terms for a binomial expansion with powers of 2, 3, and 4 to rows 2, 3, and 4 of Pascal’s triangle.

d. What do you notice?

28. Use what you have discovered to predict the binomial expansion for each of the following expressions.

$$
(a + b)^5 =
(a + b)^6 =
(a + b)^7 =
$$

29. Extension: How would you write the binomial expansion for a binomial whose terms have coefficients other than one or for binomials with negative terms? For example, how would you write the binomial expansion for $(2k + 3m)^3$? How would you write the binomial expansion for $(5x - 2y)^3$?
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Binomial Coefficients, Part 2

Binomial coefficients can also be found using the concept of combinations. The number of combinations of \( k \) objects from a set of \( n \) objects is typically written in one of several ways: \( nC_r \) or \( \binom{n}{r} \). The mathematical formula for the number of combinations is \( \frac{n!}{r!(n-r)!} \). The mathematical expression \( n! \), read “\( n \) factorial,” means “Form the product of \( n \) and all natural numbers less than \( n \).” Thus \( 4! = 4 \cdot 3 \cdot 2 \cdot 1 \). In terms of the binomial expansion, \( \binom{n}{r} \) represents the coefficient of the \( r \)th entry for an expansion to the \( n \)th power. (Note that the expansion begins with the zeroth entry. For example, for the binomial expansion of \( (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \), \( a^3 \) is the first entry; \( 3ab^2 \) is the second entry, and \( b^3 \) is the third entry.)

For Pascal’s triangle, \( n \) is the row number triangle, and \( r \) = the entry in the row. For example, \( \binom{8}{3} \) indicates we are considering a binomial expansion to the eighth power and are looking for the coefficient of the third entry. We would look in row 8 of Pascal’s triangle and find entry 3. (Recall the first number in a row is entry 0.)

30. Use Pascal’s triangle to find \( \binom{8}{3} \). Use the formula to verify your answer.

31. Use Pascal’s triangle to find the following binomial coefficients. Describe what the coefficient represents:

\[
\binom{5}{3}, \quad \binom{6}{4}, \quad \binom{9}{5}
\]

Can You …

• find real-world applications of Pascal’s triangle?
• find a relationship for the rows in which the second entry is a prime number?
• find the Fibonacci sequence in the triangle?
• find the Triangular numbers in the triangle?
• find the Square numbers in the triangle?
• find the “Hockey-Stick Rule” in the triangle?
• draw a parallelogram on Pascal’s triangle and find the sum of all numbers contained in the parallelogram? How does the sum relate to one of the numbers found outside the parallelogram? Note: The parallelogram must always contain the top “1” on Pascal’s triangle.
• use Pascal’s triangle to find the number of items given each day in the song, “The 12 Days of Christmas”?

Did You Know That …

• Pascal did not originally develop Pascal’s triangle? In 1303, Chinese mathematician Chu Shih-Chieh wrote \( Su-yuan yu-chien \), or \( Precious Mirror of the Four Elements \). He described what is now known as Pascal’s triangle and explained how it could be used to solve polynomial equations. He also invented “the method of celestial element” to write and solve polynomial equations and linear systems with up to four variables.
• the French probabilist Pierre Remond de Montmort introduced the term “Pascal’s triangle” in his 1708 book \( Essay d’Analyse sur les Jeux de Hazard \), an early application of the ideas of probability?
• if you alternate the signs of the numbers in any row and then add them together, the sum is zero?

Mathematical Content

Patterns, modular arithmetic, binomials and expansion of polynomials, combinations, factorials, reasoning and generalizing
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Bibliography


